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Non-commutative multi-solitons in 2+1 dimensions

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Abstract: The study of non-commutative solitons is greatly facilitated if the field equations are integrable, i.e. result from a linear system. For the example of a modified but integrable U(n) sigma model in $2+1$ dimensions we employ the dressing method to construct explicit multi-soliton configurations on non-commutative $\mathbb{R}^{2,1}$. These solutions, abelian and non-abelian, feature exact time-dependence for any value of the noncommutativity parameter θ and describe various lumps of finite energy in relative motion. We discuss their scattering properties and prove asymptotic factorization for large times.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry, Integrable Field Theories.

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Contents

1. Introduction

The past year has witnessed an explosion of activity in non-commutative field theory. The original motivation derived from the discovery [1] that a certain corner of string moduli space is described by non-commutative gauge theory. Since then, the field has been driven by the curiosity to extend all kinds of (quantum) field theoretical structures to the non-commutative realm [2] (for a review see [3]).

Before attempting to quantize non-commutative theories, it is certainly warranted to achieve control over the moduli space of classical configurations. Already *commutative* field theories in $1 + 1$, $2 + 1$, and $3 + 1$ dimensions display a variety of non-perturbative solutions (like solitons, vortices, monopoles, instantons) which may be interpreted as D-branes in the string context. Turning on a constant magnetic NS B-field background generalizes those branes to solitonic solutions of the *noncommutatively* deformed field theories [4, 5] (for a lecture on the subject see [6]). Since spatial noncommutativity requires at least two dimensions, most investigations have focused on scalar and gauge theories in $2+0$ and $2+1$ dimensions, where noncommutativity is tuned by a single parameter θ. One may roughly distinguish three types of results here.

Firstly, non-commutative *scalar* field theories were expanded around their $\theta \to \infty$ limit. There, the potential energy dominates, and *limiting static solutions* are given by free projectors on subspaces in a harmonic-oscillator Fock space. Corrections due to the kinetic energy can be taken into account by a perturbation series in $1/\theta$ (see [2, 7, 8, 9] and references therein). Secondly, *explicit solutions* for finite θ were considered, mainly for *abelian gauge* fields interacting with scalar matter [2] and [7]–[23]. Various properties of these configurations were investigated, including their stability under quantum fluctuations (see e.g. [15, 24, 25] and references therein). Thirdly, the *adiabatic motion* of solitons was studied by looking at the geometry of the moduli space of the *static* multi-soliton configurations (see e.g. [8, 9, 26] and references therein).

In a previous paper [27] we proposed a non-commutative generalization of an integrable sigma model with a Wess-Zumino-Witten-type term. This theory describes the dynamics of $\mathrm{U}(n)$-valued scalar fields in $2+1$ dimensions [28]. Its advantage lies in the fact that *multi-soliton* solutions to its field equations can be written down explicitly. This is possible because the equations of motion can be formulated as the compatibility conditions of some linear equations. Hence, this notion of integrability carries over to the non-commutative case, and powerful solution-generating techniques can be applied here as well.

In the present paper we describe in detail how a new solution can be constructed from an old one by a non-commutative generalization of the so-called *dressing approach* [29, 30, 31] and write down a rather general class of explicit solutions, *abelian* as well as *non-abelian*, for *any value of θ*. These configurations are parametrized by arbitrary functions, and for concrete choices we obtain (time-dependent) *multi-solitons*. The latter feature finite-energy lumps in mutually *relative motion*. We discuss their functional form, large-time asymptotics, and scattering properties.

The paper is organized as follows. We present the non-commutative extension of the modified sigma model, its energy functional and field equations in the following section. Section 3 reviews some basics on the operator formalism of non-commutative field theory. The dressing approach is outlined in section 4, where we follow our one-pole ansatz through to the general solution, which includes the introduction of a "squeezing" transformation to moving-frame coordinates. Abelian and non-abelian static solutions including their energy, BPS bound and star-product form, are discussed in section 5. We study one-soliton configurations, i.e. solutions with a single velocity parameter, in section 6, derive their energies and give examples and limits. Section 7 finally exemplifies the multi-soliton construction on the two-soliton

JHEP11(2001)040

case (with relative motion), abelian as well as non-abelian. The asymptotic behavior at high speed or large times is derived, which aids the evaluation of the energy. In three appendices, we collect formulae about coherent and squeezed states and present two different proofs of the large-time factorization of multi-soliton configurations.

2. Modified sigma model in $2+1$ dimensions

Definitions and notation. As has been known for some time, non-linear sigma models in $2+1$ dimensions may be Lorentz-invariant or integrable but not both. In this paper we choose the second property and investigate the non-commutative extension of a modified $\mathrm{U}(n)$ sigma model (so as to be integrable) introduced by Ward [28].

Classical field theory on non-commutative spaces may be realized in a star-product formulation or in an operator formalism. The first approach is closer to the commutative field theory: it is obtained by simply deforming the ordinary product of classical fields (or their components) to the non-commutative star product

$$(f \star g)(x) = f(x) \exp\left\{\frac{i}{2}\overleftarrow{\partial_a}\,\theta^{ab}\,\overrightarrow{\partial_b}\right\} g(x)\,, \tag{2.1}$$

with a constant antisymmetric tensor θ^{ab}, where $a, b, \ldots = 0, 1, 2$. Specializing to $\mathbb{R}^{1,2}$, we shall use (real) coordinates $(x^a) = (t, x, y)$ in which the Minkowskian metric reads $(\eta_{ab}) = \mathrm{diag}(-1, +1, +1)$. For later use we introduce two convenient coordinate combinations, namely

$$u := \frac{1}{2}(t+y)\,, \qquad v := \frac{1}{2}(t-y)\,, \qquad \partial_u = \partial_t + \partial_y\,, \qquad \partial_v = \partial_t - \partial_y \tag{2.2}$$

and

$$z := x + iy\,, \qquad \bar{z} := x - iy\,, \qquad \partial_z = \frac{1}{2}(\partial_x - i\partial_y)\,, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)\,. \tag{2.3}$$

Since the time coordinate remains commutative, the only non-vanishing component of the noncommutativity tensor is

$$\theta^{xy} = -\theta^{yx} =: \theta \;>\; 0 \Longrightarrow \theta^{z\bar{z}} = -\theta^{\bar{z}z} = -2i\theta\,. \tag{2.4}$$

Action and energy. The non-commutative $\mathrm{U}(n)$ sigma model may be obtained by a reduction of the Nair-Schiff sigma-model-type action [32, 33] from four to three dimensions,

$$\begin{aligned} S = &-\frac{1}{2}\int dt\,dx\,dy\;\eta^{ab}\;\mathrm{tr}\left(\partial_a\Phi^{-1}\star\partial_b\Phi\right) - \\ &-\frac{1}{3}\int dt\,dx\,dy\int_0^1 d\rho\,\widetilde{v}_\lambda\epsilon^{\lambda\mu\nu\sigma}\;\mathrm{tr}\left(\widetilde{\Phi}^{-1}\star\partial_\mu\widetilde{\Phi}\star\widetilde{\Phi}^{-1}\star\partial_\nu\widetilde{\Phi}\star\widetilde{\Phi}^{-1}\star\partial_\sigma\widetilde{\Phi}\right), \end{aligned} \tag{2.5}$$

where Greek indices include the extra coordinate ρ, and $\epsilon^{\lambda\mu\nu\sigma}$ denotes the totally antisymmetric tensor in $\mathbb{R}^4$. The field $\Phi(t,x,y)$ is group-valued, $\Phi^\dagger = \Phi^{-1}$, with an extension $\widetilde{\Phi}(t,x,y,\rho)$ interpolating between

$$\widetilde{\Phi}(t,x,y,0) = \text{const}\,, \qquad \text{and} \qquad \widetilde{\Phi}(t,x,y,1) = \Phi(t,x,y)\,, \tag{2.6}$$

and "tr" implies the trace over the $\mathrm{U}(n)$ group space. Finally, $(\widetilde{v}_\lambda) = (v_c, 0)$ is a constant vector in (extended) space-time. For $(v_c) = (0,0,0)$ one obtains the standard (Lorentz-invariant) model. Following Ward [28], we choose $(v_c) = (0,1,0)$ spacelike, which yields a modified but integrable sigma model. Although this explicitly breaks the Lorentz group of $\mathrm{SO}(1,2)$ to the $\mathrm{GL}(1,\mathbb{R})$ generated by the boost in y direction, it leaves unmodified the conserved energy functional

$$E = \frac{1}{2}\int dx\,dy\ \operatorname{tr}\left(\partial_t\Phi^\dagger\star\partial_t\Phi + \partial_x\Phi^\dagger\star\partial_x\Phi + \partial_y\Phi^\dagger\star\partial_y\Phi\right). \tag{2.7}$$

Field equations. The non-commutative sigma-model equation of motion following from (2.5) reads

$$\left(\eta^{ab} + v_c\,\epsilon^{cab}\right)\partial_a\left(\Phi^{-1}\star\partial_b\Phi\right) = 0\,, \tag{2.8}$$

where ϵ^{abc} is the alternating tensor with $\epsilon^{012} = 1$. In our coordinates (2.2) it is written more concisely as

$$\partial_x\left(\Phi^{-1}\star\partial_x\Phi\right) - \partial_v\left(\Phi^{-1}\star\partial_u\Phi\right) = 0\,. \tag{2.9}$$

This Yang-type equation [34] can be transformed into a Leznov-type equation [35],

$$\partial_x^2\phi - \partial_u\partial_v\phi + \partial_v\phi\star\partial_x\phi - \partial_x\phi\star\partial_v\phi = 0\,, \tag{2.10}$$

for an algebra-valued field $\phi(t,x,y) \in \mathrm{u}(n)$ defined via

$$\partial_x\phi := \Phi^{-1}\star\partial_u\Phi =: A\,, \qquad \text{and} \qquad \partial_v\phi := \Phi^{-1}\star\partial_x\Phi =: B\,. \tag{2.11}$$

The equation (2.9) actually arises from the Bogomolnyi equations for the $2+1$-dimensional Yang-Mills-Higgs system (A_μ, φ),

$$\frac{1}{2}\epsilon_{abc}\,F^{bc} = \partial_a\varphi + A_a\star\varphi - \varphi\star A_a\,. \tag{2.12}$$

Indeed, after choosing the gauge $A_v = 0 = A_x + \varphi$ and solving one equation by putting

$$A_u = \Phi^{-1}\star\partial_u\Phi\,, \qquad \text{and} \qquad A_x - \varphi = \Phi^{-1}\star\partial_x\Phi\,, \tag{2.13}$$

the Bogomolnyi eqs. (2.12) get reduced to (2.9).

3. Operator formalism

Fock space. The nonlocality of the star product renders explicit computations cumbersome. We therefore pass to the operator formalism, which trades the star product for operator-valued coordinates $\hat{x}^\mu$ satisfying $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$. The noncommutative coordinates for $\mathbb{R}^{1,2}$ are $(t, \hat{x}, \hat{y})$ subject to

$$[t, \hat{x}] = [t, \hat{y}] = 0 \,, \qquad [\hat{x}, \hat{y}] = i\theta \longrightarrow \left[\hat{z}, \hat{\bar{z}}\right] = 2\theta \,. \tag{3.1}$$

The latter equation suggests the introduction of (properly normalized) creation and annihilation operators,

$$a = \frac{1}{\sqrt{2\theta}}\,\hat{z}\,, \qquad \text{and} \qquad a^\dagger = \frac{1}{\sqrt{2\theta}}\,\hat{\bar{z}}\,, \qquad \text{so that} \qquad \left[a, a^\dagger\right] = 1\,. \tag{3.2}$$

They act on a harmonic-oscillator Fock space $\mathcal{H}$ with an orthonormal basis $\{|n\rangle,\ n = 0, 1, 2, \ldots\}$ such that

$$a^\dagger a\,|n\rangle =: N\,|n\rangle = n\,|n\rangle\,, \qquad a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{3.3}$$

Moyal-Weyl map. Any function $f(t, z, \bar{z})$ can be related to an operator-valued function $\hat{f}(t) \equiv F(t, a, a^\dagger)$ acting in $\mathcal{H}$, with the help of the Moyal-Weyl map (see e.g. [24, 36])

$$\begin{aligned} f(t, z, \bar{z}) \longrightarrow F\left(t, a, a^\dagger\right) &= -\int \frac{dp\, d\bar{p}}{(2\pi)^2}\, dz\, d\bar{z}\; f(t, z, \bar{z})\; e^{-i[\bar{p}(\sqrt{2\theta}a - z) + p(\sqrt{2\theta}a^\dagger - \bar{z})]} \\ &= \text{Weyl-ordered } f\left(t, \sqrt{2\theta}\,a, \sqrt{2\theta}\,a^\dagger\right). \end{aligned} \tag{3.4}$$

The inverse transformation recovers the c-number function,

$$\begin{aligned} F\left(t, a, a^\dagger\right) \longrightarrow f(t, z, \bar{z}) &= 2\pi\theta \int \frac{2i\, dp\, d\bar{p}}{(2\pi)^2}\ \mathrm{Tr}\left\{F\left(t, a, a^\dagger\right) e^{i[\bar{p}(\sqrt{2\theta}a - z) + p(\sqrt{2\theta}a^\dagger - \bar{z})]}\right\} \\ &= F_\star\left(t, \frac{z}{\sqrt{2\theta}}, \frac{\bar{z}}{\sqrt{2\theta}}\right), \end{aligned} \tag{3.5}$$

where "Tr" signifies the trace over the Fock space $\mathcal{H}$, and $F_\star$ is obtained from F by replacing ordinary with star products. Under the Moyal-Weyl map, we have

$$f \star g \longrightarrow \hat{f}\,\hat{g}\,, \qquad \text{and} \qquad \int dx\, dy\, f = 2\pi\theta\ \mathrm{Tr}\ \hat{f} = 2\pi\theta \sum_{n\geq 0} \langle n|\hat{f}|n\rangle. \tag{3.6}$$

The operator formulation turns spatial derivatives into commutators,

$$\partial_x f \longrightarrow \frac{i}{\theta}\left[\hat{y}, \hat{f}\right] =: \hat{\partial}_x \hat{f}\,, \qquad \text{and} \qquad \partial_y f \longrightarrow -\frac{i}{\theta}\left[\hat{x}, \hat{f}\right] =: \hat{\partial}_y \hat{f}\,, \tag{3.7}$$

so that

$$\partial_z f \longrightarrow \hat{\partial}_z \hat{f} = \frac{-1}{\sqrt{2\theta}} \left[a^\dagger, \hat{f}\right], \qquad \text{and} \qquad \partial_{\bar{z}} f \longrightarrow \hat{\partial}_{\bar{z}} \hat{f} = \frac{1}{\sqrt{2\theta}} \left[a, \hat{f}\right]. \tag{3.8}$$

The basic examples for the relation between f and $\hat{f}$ are

$$\begin{array}{llllllll} f: & 1 & z & \bar{z} & z\bar{z}-\theta & z\bar{z} & z\bar{z}+\theta & , \\ \hat{f}: & 1 & \sqrt{2\theta}\,a & \sqrt{2\theta}\,a^\dagger & 2\theta N & 2\theta\left(N+\frac{1}{2}\right) & 2\theta(N+1) & . \end{array} \tag{3.9}$$

For more complicated functions it helps to remember that

$$f \star f = f^2, \qquad z \star \bar{z} = z\bar{z} + \theta, \qquad \bar{z} \star z = z\bar{z} - \theta, \tag{3.10}$$

because the composition law (3.6) allows one to employ the star product (on the functional side) instead of the Weyl ordering (on the operator side). For notational simplicity we will from now on omit the hats over the operators except when confusion may arise.

4. Dressing approach and explicit solutions

The payoff for considering an integrable model is the availability of powerful techniques for constructing solutions to the equation of motion. One of these tools is the so-called "dressing method", which was invented to generate solutions for commutative integrable systems [29, 30, 31] and is easily extended to the non-commutative setup [27]. Let us briefly present this method (already in the non-commutative context) before applying it to the modified sigma model.

Linear system. We consider the two linear equations

$$(\zeta\partial_x - \partial_u)\psi = A\,\psi, \qquad \text{and} \qquad (\zeta\partial_v - \partial_x)\psi = B\,\psi, \tag{4.1}$$

which can be obtained from the Lax pair for the (non-commutative) self-dual Yang-Mills equations in $\mathbb{R}^{2,2}$ [37, 38] by gauge-fixing and imposing the condition $\partial_3\psi = 0$. Here, ψ depends on (t, x, y, ζ) or, equivalently, on (x, u, v, ζ) and is an $n \times n$ matrix whose elements act as operators in the Fock space $\mathcal{H}$. The matrices A and B are of the same type as ψ but do not depend on ζ. The spectral parameter ζ lies in the extended complex plane. The matrix ψ is subject to the following reality condition [28]:

$$\psi(t, x, y, \zeta)\left[\psi\left(t, x, y, \bar{\zeta}\right)\right]^\dagger = 1, \tag{4.2}$$

where "†" is hermitean conjugation. We also impose on ψ the standard asymptotic conditions [38]

$$\begin{aligned} \psi(t, x, y, \zeta \longrightarrow \infty) &= 1 + \zeta^{-1}\phi(t, x, y) + O\left(\zeta^{-2}\right), \\ \psi(t, x, y, \zeta \longrightarrow 0) &= \Phi^{-1}(t, x, y) + O(\zeta). \end{aligned} \tag{4.3}$$

The compatibility conditions for the linear system of differential eqs. (4.1) read

$$\begin{aligned} \partial_x B - \partial_v A &= 0\,, \\ \partial_x A - \partial_u B - [A, B] &= 0\,. \end{aligned} \tag{4.4}$$

We have already encountered "solutions" of these equations: expressing A and B in terms of ϕ like in (2.11),

$$A = \partial_x \phi = \Phi^{-1}\,\partial_u \Phi\,, \qquad \text{and} \qquad B = \partial_v \phi = \Phi^{-1}\,\partial_x \Phi\,, \tag{4.5}$$

solves the first equation and turns the second into (the operator form of) our Leznov-type eq. (2.10),

$$\partial_x^2 \phi - \partial_u \partial_v \phi - [\partial_x \phi\,,\, \partial_v \phi] = 0\,. \tag{4.6}$$

Alternatively, the ansatz employing Φ in (4.5) fulfils the second equation and transforms the first one into (the operator version of) our Yang-type eq. (2.9),

$$\partial_x \left(\Phi^{-1}\,\partial_x \Phi\right) - \partial_v \left(\Phi^{-1}\,\partial_u \Phi\right) = 0\,. \tag{4.7}$$

Inserting these two parametrizations of A and B into the linear system (4.1) we immediately verify (4.3), confirming that the identical notation for A, B, ϕ, and Φ in section 2 and in the present one was justified.

Dressing approach and ansatz. Having identified auxiliary linear first-order differential equations pertaining to our second-order non-linear equation, we set out to solve the former. Note that the knowledge of ψ yields ϕ and Φ by way of (4.3), respectively, and thus A and B via (4.5) or directly from

$$\begin{aligned} -\psi(t,x,y,\zeta)\,(\zeta\partial_x - \partial_u)\,\left[\psi\left(t,x,y,\bar\zeta\,\right)\right]^\dagger &= A(t,x,y)\,, \\ -\psi(t,x,y,\zeta)\,(\zeta\partial_v - \partial_x)\,\left[\psi\left(t,x,y,\bar\zeta\,\right)\right]^\dagger &= B(t,x,y)\,. \end{aligned} \tag{4.8}$$

The dressing method is a recursive procedure generating a new solution from an old one. Let us have a solution $\psi_0(t,x,y,\zeta)$ of the linear eqs. (4.1) for a given solution (A_0, B_0) of the field eqs. (4.4). Then we can look for a new solution ψ in the form

$$\begin{aligned} \psi(t,x,y,\zeta) &= \chi(t,x,y,\zeta)\psi_0(t,x,y,\zeta)\,, \\ \text{with} \qquad \chi &= 1 + \sum_{\alpha=1}^{s}\sum_{k=1}^{m} \frac{R_{k\alpha}}{(\zeta-\mu_k)^\alpha}\,, \end{aligned} \tag{4.9}$$

where the $\mu_k(t,x,y)$ are complex functions and the $n\times n$ matrices $R_{k\alpha}(t,x,y)$ are independent of ζ.

In this paper we shall not discuss this approach in full generality (for more details see e.g. [29, 30, 31]). We consider the vacuum seed solution $A_0 = B_0 = 0$, $\psi_0 = 1$ and a restricted ansatz containing only first-order poles in ζ:

$$\psi = \chi\,\psi_0 = 1 + \sum_{p=1}^{m} \frac{R_p}{\zeta - \mu_p}\,, \tag{4.10}$$

JHEP11(2001)040

where the μ_p are complex constants with $\operatorname{Im}\mu_k < 0$. Moreover, following [31], we take the matrices R_k to be of the form

$$R_k = \sum_{\ell=1}^{m} T_\ell\, \Gamma^{\ell k}\, T_k^\dagger \,, \tag{4.11}$$

where the $T_k(t,x,y)$ are $n \times r$ matrices and $\Gamma^{\ell k}(t,x,y)$ are $r \times r$ matrices with some $r \geq 1$. In the abelian case ($n = 1$) one may think of T_k as a time-dependent row vector $(|z_k^1,t\rangle, |z_k^2,t\rangle, \ldots, |z_k^r,t\rangle)$ of (ket) states in $\mathcal{H}$.

We must make sure to satisfy the reality condition (4.2) as well as our linear equations in the form (4.8). In particular, the poles at $\zeta = \bar{\mu}_k$ on the l.h.s. of these equations have to be removable since the r.h.s. are independent of ζ. Inserting the ansatz (4.10) with (4.11) and putting to zero the corresponding residues, we learn from (4.2) that

$$\left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k}\right) T_k = 0 \tag{4.12}$$

while from (4.8) we obtain the differential equations

$$(A \text{ or } B) = -\psi(\zeta)\, \overline{L}_k\, \psi\left(\bar{\zeta}\right)^\dagger |_{\zeta\to\bar{\mu}_k} \implies \left(1 - \sum_{p=1}^{m} \frac{R_p}{\mu_p - \bar{\mu}_k}\right) \overline{L}_k\, R_k^\dagger = 0\,, \tag{4.13}$$

where

$$\overline{L}_k := \bar{\mu}_k \partial_x - \partial_u \,, \qquad \text{or} \qquad \overline{L}_k := \bar{\mu}_k \partial_v - \partial_x \,. \tag{4.14}$$

The algebraic conditions (4.12) imply that the $\Gamma^{\ell p}$ invert the matrices

$$\widetilde{\Gamma}_{pk} = \frac{1}{\mu_p - \bar{\mu}_k}\, T_p^\dagger\, T_k \,, \qquad \text{i.e.} \qquad \sum_{p=1}^{m} \Gamma^{\ell p}\, \widetilde{\Gamma}_{pk} = \delta^\ell{}_k \,. \tag{4.15}$$

Finally, by substituting (4.10) and (4.11) into the formulae (4.3) we solve the equations of motion (4.6) and (4.7) by

$$\Phi^{-1} = \Phi^\dagger = 1 - \sum_{k,\ell=1}^{m} \frac{1}{\mu_k}\, T_\ell\, \Gamma^{\ell k}\, T_k^\dagger \,, \qquad \text{and} \qquad \phi = \sum_{k,\ell=1}^{m} T_\ell\, \Gamma^{\ell k}\, T_k^\dagger \tag{4.16}$$

for every solution to (4.12) and (4.13).

Moving-frame coordinates. At this point it is a good idea to introduce the co-moving coordinates

$$\begin{aligned} w_k &:= \nu_k \left[x + \bar{\mu}_k u + \bar{\mu}_k^{-1} v\right] = \nu_k \left[x + \frac{1}{2}\left(\bar{\mu}_k - \bar{\mu}_k^{-1}\right) y + \frac{1}{2}\left(\bar{\mu}_k + \bar{\mu}_k^{-1}\right) t\right], \\ \overline{w}_k &:= \bar{\nu}_k \left[x + \mu_k u + \mu_k^{-1} v\right] = \bar{\nu}_k \left[x + \frac{1}{2}\left(\mu_k - \mu_k^{-1}\right) y + \frac{1}{2}\left(\mu_k + \mu_k^{-1}\right) t\right], \end{aligned} \tag{4.17}$$

where the ν_k are functions of μ_k via

$$\nu_k = \left[\frac{4i}{\mu_k - \bar{\mu}_k - \mu_k^{-1} + \bar{\mu}_k^{-1}} \cdot \frac{\mu_k - \mu_k^{-1} - 2i}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2i}\right]^{1/2} . \tag{4.18}$$

In terms of the moving-frame coordinates the linear operators defined in (4.14) become

$$\overline{L}_k = \bar{\nu}_k(\bar{\mu}_k - \mu_k)\partial_{\overline{w}_k} , \qquad \text{or} \qquad \overline{L}_k = \bar{\nu}_k\, \mu_k^{-1}(\bar{\mu}_k - \mu_k)\partial_{\overline{w}_k} , \tag{4.19}$$

respectively, so that we have only one eq. (4.13) for each pole. Since

$$\left[\hat{w}_k, \hat{\overline{w}}_k\right] = \frac{i}{2}\theta\, \nu_k\bar{\nu}_k\left(\mu_k - \bar{\mu}_k - \mu_k^{-1} + \bar{\mu}_k^{-1}\right) = 2\,\theta \tag{4.20}$$

we are led to the definition of co-moving creation and annihilation operators

$$c_k = \frac{1}{\sqrt{2\theta}}\,\hat{w}_k , \qquad \text{and} \qquad c_k^\dagger = \frac{1}{\sqrt{2\theta}}\,\hat{\overline{w}}_k , \qquad \text{so that} \qquad \left[c_k, c_k^\dagger\right] = 1 . \tag{4.21}$$

The static case ($w_k \equiv z$) is recovered for $\mu_k = -i$. Coordinate derivatives are represented in the standard fashion as [24, 36]

$$\sqrt{2\theta}\,\partial_{w_k} \longrightarrow -\left[c_k^\dagger, .\right] , \qquad \sqrt{2\theta}\,\partial_{\overline{w}_k} \longrightarrow \left[c_k, .\right] . \tag{4.22}$$

The number operator $N_k := c_k^\dagger c_k$ is diagonal in a co-moving oscillator basis $\{|n\rangle_k, n = 0, 1, 2, \ldots\}$.

It is essential to relate the co-moving oscillators to the static one. Expressing w_k by z via (4.17) and (4.18) and using (3.2) one gets

$$c_k = (\cosh\tau_k)\,a - \left(e^{i\vartheta_k}\sinh\tau_k\right)a^\dagger - \beta_k\,t = U_k(t)\,a\,U_k^\dagger(t) , \tag{4.23}$$

which is an inhomogeneous SU$(1,1)$ transformation mediated by the operator [39]

$$U_k(t) = e^{\frac{1}{2}\alpha_k\,a^{\dagger 2} - \frac{1}{2}\bar{\alpha}_k\,a^2}\; e^{(\beta_k\,a^\dagger - \bar{\beta}_k\,a)t} . \tag{4.24}$$

Here, we have introduced the complex quantities

$$\alpha_k = e^{i\vartheta_k}\,\tau_k , \qquad \text{and} \qquad \beta_k = -\frac{1}{2}\nu_k\left(\bar{\mu}_k + \bar{\mu}_k^{-1}\right)/\sqrt{2\theta} , \tag{4.25}$$

with the relations

$$\xi_k := e^{i\vartheta_k}\tanh\tau_k = \frac{\bar{\mu}_k - \bar{\mu}_k^{-1} - 2i}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2i} , \qquad \text{and} \qquad \nu_k = \cosh\tau_k - e^{i\vartheta_k}\sinh\tau_k , \tag{4.26}$$

so that μ_k may be expressed in terms of ξ_k via $\bar{\mu}_k = i\frac{(1+\sqrt{\xi_k})^2}{1-\xi_k}$. Clearly, we have $|n\rangle_k = U_k(t)|n\rangle$, and all co-moving oscillators are unitarily equivalent to the static one. The unitary transformation (4.23) is known as "squeezing" (by α_k) plus "shifting" (by $\beta_k\,t$).

Solutions. After this diversion, the remaining equations of motion (4.13) take the form

$$\left(1-\sum_{p=1}^{m}\frac{R_p}{\mu_p-\bar{\mu}_k}\right)c_k\,T_k=0\,, \tag{4.27}$$

which means that c_kT_k lies in the kernel of the parenthetical expression. Obviously, a sufficient condition for a solution is

$$c_k\,T_k=T_k\,Z_k \tag{4.28}$$

with some $r\times r$ matrix Z_k. We briefly elaborate on two important special cases.

Firstly, if $n\geq 2$ and

$$Z_k=\mathbf{1}\,c_k\,,\qquad \text{then}\qquad [c_k,T_k]=0\,, \tag{4.29}$$

which can be interpreted as a holomorphicity condition on T_k. Hence, any $n\times r$ matrix T_k whose entries are arbitrary functions of c_k (but independent of $c_k^\dagger$) provides a solution ψ of the linear system (4.1), after inserting it into (4.10), (4.11) and (4.15). The complex moduli μ_k turn out to parametrize the velocities $\vec{v}_k$ of individual asymptotic lumps of energy. The shape and location of these lumps is encoded in further moduli hiding in the functions $T_k(c_k)$.

Secondly, the abelian case ($n=1$) deserves some attention. Here, the entries of the row vector $T_k=(|z_k^1,t\rangle,|z_k^2,t\rangle,\ldots,|z_k^r,t\rangle)$ are *not* functions of c_k. If we take

$$\begin{aligned} Z_k&=\mathrm{diag}\left(z_k^1,z_k^2,\ldots,z_k^r\right)\,,\qquad \text{with}\qquad z_k^i\in\mathbb{C}\,,\\ \text{then}\qquad c_k\left|z_k^i,t\right\rangle&=z_k^i\left|z_k^i,t\right\rangle\,, \end{aligned} \tag{4.30}$$

which explains our labeling of the kets and qualifies them as coherent states based on the co-moving ground state $|0\rangle_k$,

$$\left|z_k^i,t\right\rangle=e^{z_k^ic_k^\dagger-\bar{z}_k^ic_k}|0\rangle_k=e^{z_k^ic_k^\dagger-\bar{z}_k^ic_k}\,U_k(t)\,|0\rangle=U_k(t)\,e^{z_k^ia^\dagger-\bar{z}_k^ia}\,|0\rangle\,. \tag{4.31}$$

In total, the abelian solutions $|z_k^i,t\rangle$ are nothing but squeezed states. The squeezing parameter is α_k while the (complex) shift parameter is $z_k^i+\beta_kt$. In the star-product picture, we find m groups of r asymptotic lumps of energy. The squeezing parametrizes the (common) deviation of the lumps of type k from spherical shape, while the shifts yield the position of each lump in the non-commutative plane. Clearly, all lumps in one group move with equal constant velocity but are arbitrarily separated in the plane. This discussion extends naturally to the non-abelian case.

We remark that our solutions have a four-dimensional perspective. Recall that the linear system (4.1) can be produced from the linear system of the self-dual Yang-Mills equations on non-commutative $\mathbb{R}^{2,2}$. Therefore, the dressing approach may be employed in this more general situation to create BPS-type solutions of the functional form (4.16) to the self-dual Yang-Mills equations in $2+2$ dimensions. Hence, our soliton solutions of (4.7) in $2+1$ dimensions can be obtained from these infinite-action BPS-type solutions by dimensional reduction. In contrast, BPS-type solutions of the form (4.16) do not exist on non-commutative $\mathbb{R}^{4,0}$ because the reality condition (4.2) for the matrix ψ then involves $\zeta \to -\frac{1}{\zeta}$ instead of $\zeta \to \bar{\zeta}$. However, in four euclidean dimensions a modified ADHM approach allows one to construct non-commutative instanton solutions the first examples of which were given by Nekrasov and Schwarz [40].

5. Static solutions

Projectors. For comparison with earlier work, let us consider the static case, $m = 1$ and $\mu = -i$. Staying with our restricted ansatz of first-order poles only, the expression (4.10) then simplifies to

$$\psi = 1 + \frac{R}{\zeta + i} =: 1 - \frac{2i}{\zeta + i}\, P\,, \qquad \text{so that} \qquad \Phi^\dagger = 1 - 2\,P = P_\perp - P\,. \tag{5.1}$$

The reality condition (4.2) directly yields

$$P^\dagger = P\,, \qquad \text{and} \qquad P^2 = P\,, \tag{5.2}$$

qualifying P as a hermitean projector. Indeed, (4.2) degenerates to $(1-P)P = 0$, while (4.13) simplifies to

$$(1-P)\,\overline{L}\,P = 0 \Longrightarrow F := (1-P)\,a\,P = 0\,. \tag{5.3}$$

We emphasize that this equation also characterizes the BPS subsector of general non-commutative scalar field theories [8, 9].

It is illuminating to specialize also our ansatz (4.11) for the residue, with (4.15), to the static situation:

$$\Gamma = \frac{-2i}{T^\dagger T} \Longrightarrow P \;\equiv\; \frac{R}{-2i} = T\,\frac{\Gamma}{-2i}\,T^\dagger = T\,\frac{1}{T^\dagger T}\,T^\dagger\,. \tag{5.4}$$

Hence, the projector P is parametrized by an $n \times r$ matrix T, where, in the non-abelian case, $r \leq n$ can be identified with the rank of the projector in the $\mathrm{U}(n)$ group space (but not in $\mathcal{H}$). The equation (4.27) then simplifies to

$$(1-P)\,a\,T = 0\,, \tag{5.5}$$

which means that aT must lie in the kernel of $1-P$.

JHEP11(2001)040

Energy and BPS bound. Clearly, static configurations are real, $\Phi^\dagger = \Phi$. The energy (2.7) of solutions to (5.3) reduces to

$$\begin{aligned} E \;&= 4\pi\theta\ \mathrm{Tr}\left(\hat\partial_z\Phi\,\hat\partial_{\bar z}\Phi\right) \\ &= 8\pi\ \mathrm{Tr}\left(\left[a^\dagger\,,P\right]\left[P\,,\,a\right]\right) = 8\pi\ \mathrm{Tr}\left(a^\dagger P a - P\,a^\dagger a + 2a^\dagger F\right) \\ &= 8\pi\ \mathrm{Tr}\left(a^\dagger P a - P\,a^\dagger a\right) \overset{?}{=}\ 8\pi\ \mathrm{Tr}\ P\,, \end{aligned} \tag{5.6}$$

where the last equality holds only for projectors of *finite* rank in $\mathcal{H}$.

The topological charge Q derives from a BPS argument. The energy of a general static configuration (not necessarily a solution of $F=0$) obeys

$$\begin{aligned} \frac{1}{8\pi}\,E \;&= \mathrm{Tr}\left(\left[a^\dagger,P\right]\left[P\,,a\right]\right) \\ &= \mathrm{Tr}\left(P\left[a^\dagger,P\right]\left[a\,,P\right] - P\left[a\,,P\right]\left[a^\dagger,P\right]\right) + \mathrm{Tr}\left(F^\dagger F + F\,F^\dagger\right) \\ &\geq \mathrm{Tr}\left(P\left[a^\dagger,P\right]\left[a\,,P\right] - P\left[a\,,P\right]\left[a^\dagger,P\right]\right) \\ &= \frac{i}{8}\,\theta\ \mathrm{Tr}\left(\Phi\ \hat\partial_x\Phi\ \hat\partial_y\Phi - \Phi\ \hat\partial_y\Phi\ \hat\partial_x\Phi\right) =:\, Q\,. \end{aligned} \tag{5.7}$$

Alternatively, using $G := (1-P)\,a^\dagger\,P$ in

$$\begin{aligned} \frac{1}{8\pi}\,E \;&= \mathrm{Tr}\left(\left[a^\dagger,P\right]\left[P\,,a\right]\right) \\ &= -\,\mathrm{Tr}\left(P\left[a^\dagger,P\right]\left[a\,,P\right] - P\left[a\,,P\right]\left[a^\dagger,P\right]\right) + \mathrm{Tr}\left(G^\dagger G + G\,G^\dagger\right) \end{aligned} \tag{5.8}$$

yields, along the same line, $E \geq -8\pi\,Q$. In combination, we get

$$E \geq 8\pi\,|Q|\,, \qquad \text{with} \qquad E = 8\pi\,|Q|\,, \qquad \text{iff} \qquad F=0\,, \qquad \text{or} \qquad G=0\,. \tag{5.9}$$

The $F=0$ configurations carry positive topological charge (solitons) whereas the $G=0$ solutions come with negative values of Q (antisolitons).

It is instructive to imbed the BPS sector into the complete sigma-model configuration space. For static configurations, the sigma-model eq. (4.7) becomes

$$\partial_z\left(\Phi^\dagger\,\partial_{\bar z}\Phi\right) + \partial_{\bar z}\left(\Phi^\dagger\,\partial_z\Phi\right) = 0 \overset{\Phi=1-2P}{\Longrightarrow} \left[\left[a,\left[a^\dagger,P\right]\right],P\right] = 0\,. \tag{5.10}$$

We emphasize that this is the equation of motion for the standard non-commutative euclidean two-dimensional sigma model, as was clarified previously [27]. A short calculation shows that solitons ($F=0$) as well as antisolitons ($G=0$) satisfy this equation. To summarize, our choice of ansatz for ψ leads to the construction of solitons. For describing antisolitons it suffices to switch from μ_k to $\bar\mu_k$.

Abelian static solutions. Abelian solutions ($n = 1$) are now generated at ease. Since $\mu = -i$ implies $\alpha = 0$ and $\beta = 0$, the squeezing disappears and r lumps are simply sitting at fixed positions z^i, with $i = 1, \ldots, r$. Hence, we have

$$T = \left(\left|z^1\right\rangle, \left|z^2\right\rangle, \ldots, \left|z^r\right\rangle\right), \qquad \text{with} \qquad \left|z^i\right\rangle = e^{z^i a^\dagger - \bar{z}^i a}\left|0\right\rangle. \tag{5.11}$$

This is obvious because $w \to z$ in the static case. The coherent states $|z^i\rangle$ are normalized to unity but are in general not orthogonal. One reads off the rank r projector

$$P = \sum_{i,j=1}^{r} \left|z^i\right\rangle \frac{1}{\left\langle z^i | z^j \right\rangle} \left\langle z^j\right| = \mathcal{U} \sum_{n=0}^{r-1} |n\rangle\langle n|\, \mathcal{U}^\dagger, \tag{5.12}$$

where the last equality involving a unitary operator $\mathcal{U}$ was shown in [9]. Clearly, the row vector $T = (|0\rangle, |1\rangle, \ldots, |r-1\rangle)$ directly solves (5.5), and P projects onto the space spanned by the r lowest oscillator states. Obviously, these solutions have $E = 8\pi r$.

Non-abelian static solutions. Another special case are non-abelian solutions with $r = 1$. They are provided by column vectors T whose entries are functions of a only. If we choose polynomials of at maximal degree q, the energy of the configuration will be $E = 8\pi q$, independently of the gauge group. To illustrate this fact we sketch the following U(2) example:

$$T = \begin{pmatrix} \lambda \\ a^q \end{pmatrix} \Longrightarrow P = \begin{pmatrix} \frac{\lambda\bar{\lambda}}{\lambda\bar{\lambda}+N!/(N-q)!} & \frac{\lambda}{\lambda\bar{\lambda}+N!/(N-q)!}\, a^{\dagger q} \\ a^q \, \frac{\bar{\lambda}}{\lambda\bar{\lambda}+N!/(N-q)!} & a^q \, \frac{1}{\lambda\bar{\lambda}+N!/(N-q)!}\, a^{\dagger q} \end{pmatrix}, \tag{5.13}$$

which is of infinite rank in $\mathcal{H}$. Still, the energy (5.6) is readily computed with the result of $8\pi q$.

Inverse Moyal-Weyl map. Having constructed non-commutative solitons in operator language, it is natural to wonder how these configurations look like in the star-product formulation. The back translation to functions $\Phi_\star(t, x, y)$ solving the non-commutative field eqs. (2.9) is easily accomplished by the inverse Moyal-Weyl map (3.5). We take a brief look at $r = 1$ solutions.

The abelian case has been considered many times in the literature [2, 6, 15, 16]. Here one has

$$\Phi = 1 - 2\,|0\rangle\langle 0| \longrightarrow \Phi_\star = 1 - 4\, e^{-z\bar{z}/\theta}, \tag{5.14}$$

which lacks a non-trivial $\theta \to 0$ or $\theta \to \infty$ limit.

For a non-abelian example we consider the static U(2) solution $\Phi = 1 - 2P$ with P given by (5.13), for degree $q = 1$. With the methods outlined in section 3, we find

$$\begin{aligned} \Phi &= 1 - 2 \begin{pmatrix} \frac{\lambda\bar{\lambda}}{\lambda\bar{\lambda}+N} & \frac{\lambda}{\lambda\bar{\lambda}+N} a^\dagger \\ a \frac{\bar{\lambda}}{\lambda\bar{\lambda}+N} & a \frac{1}{\lambda\bar{\lambda}+N} a^\dagger \end{pmatrix} \\ \longrightarrow \Phi_\star &= \begin{pmatrix} -\frac{\gamma\bar{\gamma}-z\bar{z}+\theta}{\gamma\bar{\gamma}+z\bar{z}-\theta} & -2\gamma \frac{\gamma\bar{\gamma}+z\bar{z}-2\theta}{(\gamma\bar{\gamma}+z\bar{z}-\theta)^2} \bar{z} \\ -2z \frac{\gamma\bar{\gamma}+z\bar{z}-2\theta}{(\gamma\bar{\gamma}+z\bar{z}-\theta)^2} \bar{\gamma} & +\frac{\gamma\bar{\gamma}-z\bar{z}-\theta}{\gamma\bar{\gamma}+z\bar{z}+\theta} \end{pmatrix} \end{aligned} \tag{5.15}$$

with $\gamma = \sqrt{2\theta}\lambda$. One may check that it indeed fulfils (2.9).

JHEP11(2001)040

It is instructive to display the $\theta \to 0$ and $\theta \to \infty$ limits of $\Phi_\star$, while keeping γ fixed:

$$\Phi_\star(\theta \longrightarrow 0) \longrightarrow \begin{pmatrix} -\frac{\gamma\overline{\gamma}-z\overline{z}}{\gamma\overline{\gamma}+z\overline{z}} & \frac{-2\gamma\overline{z}}{\gamma\overline{\gamma}+z\overline{z}} \\ \frac{-2z\overline{\gamma}}{\gamma\overline{\gamma}+z\overline{z}} & +\frac{\gamma\overline{\gamma}-z\overline{z}}{\gamma\overline{\gamma}+z\overline{z}} \end{pmatrix},$$

$$\text{and} \qquad \Phi_\star(\theta \longrightarrow \infty) \longrightarrow \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.16}$$

Other than the abelian solution, $\Phi_\star(z, \overline{z}, \theta)$ is a rational function with a non-trivial commutative limit.

6. One-soliton configurations

For a time-dependent configuration with $m = 1$, our first-order pole ansatz (4.10) simplifies to

$$\psi = 1 + \frac{R}{\zeta - \mu} =: 1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} P, \qquad \text{with} \qquad P = T \frac{1}{T^\dagger T} T^\dagger, \tag{6.1}$$

where P and T satisfy

$$(1 - P)\, c\, P = 0, \qquad \text{and} \qquad c\, T = T\, Z. \tag{6.2}$$

Here,

$$c = (\cosh\tau)\, a - \left(e^{i\vartheta} \sinh\tau\right) a^\dagger - \beta\, t = U(t)\, a\, U^\dagger(t) \tag{6.3}$$

is the creation operator in the moving frame.

Comparing the formulae of section 4 with (6.1), (6.2) and (6.3) we see that the operators $c, c^\dagger$ and therefore the matrix T and the projector P in (6.2) can be expressed in terms of the corresponding static objects (see section 5). This is accomplished by means of the inhomogeneous SU(1,1) transformation (6.3), which on the coordinates reads

$$\begin{pmatrix} w \\ \overline{w} \end{pmatrix} = \begin{pmatrix} \cosh\tau & -e^{i\vartheta}\sinh\tau \\ -e^{-i\vartheta}\sinh\tau & \cosh\tau \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \nu\,(\bar{\mu} + \bar{\mu}^{-1}) \\ \bar{\nu}\,(\mu + \mu^{-1}) \end{pmatrix} t \tag{6.4}$$

and is in fact a symplectic coordinate transformation, preserving $dz \wedge d\overline{z}$. Even though the inhomogeneous part of (6.4) is linear in time, this does *not* mean that the solution

$$\Phi \equiv \Phi_{\vec{v}} = 1 - \bar{\rho}\, P, \qquad \text{with} \qquad \rho = 1 - \bar{\mu}/\mu \tag{6.5}$$

is simply obtained by subjecting the corresponding static solution $\Phi_{\vec{0}}$ to such a "symplectic boost". To see this, consider the moving frame with the coordinates $w, \overline{w}$ and $t' = t$ and the related change of derivatives,

$$\begin{aligned} \partial_x &= \nu\, \partial_w + \bar{\nu}\, \partial_{\overline{w}}, \\ \partial_t &= \frac{1}{2}\nu\left(\bar{\mu} + \bar{\mu}^{-1}\right)\partial_w + \frac{1}{2}\bar{\nu}\left(\mu + \mu^{-1}\right)\partial_{\overline{w}} + \partial_{t'}, \\ \partial_y &= \frac{1}{2}\nu\left(\bar{\mu} - \bar{\mu}^{-1}\right)\partial_w + \frac{1}{2}\bar{\nu}\left(\mu - \mu^{-1}\right)\partial_{\overline{w}}. \end{aligned} \tag{6.6}$$

In the moving frame our solution (6.5) will be static, i.e. $\partial_{t'}\Phi = 0$. However, the field eq. (4.7) is *not* invariant w.r.t. the coordinate transformation (6.4). Therefore, the solution (6.5), which is static in the moving frame, has a functional form different from that of the static solution (5.1) — the coefficient of the projector is altered from 2 to $\bar\rho$. Due to this non-invariance, the coordinate frame $(z,\overline{z},t)$ is distinguished because in it the field equation takes the simplest form.

The constant velocity $\vec{v}$ in the xy plane is easily derived by writing down the map (6.4) in real coordinates ($w =: x' + iy'$),[2]

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh\tau - \cos\vartheta\sinh\tau & -\sin\vartheta\sinh\tau \\ -\sin\vartheta\sinh\tau & \cosh\tau + \cos\vartheta\sinh\tau \end{pmatrix} \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} v_x \\ v_y \end{pmatrix} t \right], \tag{6.7}$$

where we find

$$\vec{v} \equiv (v_x, v_y) = -\left(\frac{\cos\varphi}{\cosh\eta}\,,\,\frac{\sinh\eta}{\cosh\eta}\right), \qquad \text{for} \qquad \mu =: e^{\eta - i\varphi}\,. \tag{6.8}$$

The velocity dependence of the energy

$$E = \pi\theta\ \mathrm{Tr}\left(g^{ab}\,\partial_a\Phi_{\vec{v}}^\dagger\,\partial_b\Phi_{\vec{v}}\right), \qquad \text{with} \qquad \left(g^{ab}\right) = \mathrm{diag}\,(+1,+1,+1) \tag{6.9}$$

is now obtained straightforwardly by employing derivatives w.r.t. w and $\overline{w}$ via (6.6), which yields

$$\begin{pmatrix} g^{w\overline{w}} & g^{ww} \\ g^{\overline{w}\overline{w}} & g^{\overline{w}w} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \nu\bar\nu\left(\sqrt{\mu\bar\mu} + \sqrt{\mu\bar\mu}^{-1}\right)^2 & \nu^2(\bar\mu + \bar\mu^{-1})^2 \\ \bar\nu^2(\mu + \mu^{-1})^2 & \nu\bar\nu\left(\sqrt{\mu\bar\mu} + \sqrt{\mu\bar\mu}^{-1}\right)^2 \end{pmatrix}. \tag{6.10}$$

Inserting this bilinear form and (6.5) into the energy functional (6.9), we are left with [27]

$$\begin{aligned} E[\Phi_{\vec{v}}] &= \pi\theta\,\nu\bar\nu\left(\sqrt{\mu\bar\mu} + \sqrt{\mu\bar\mu}^{-1}\right)^2 \mathrm{Tr}\left(\partial_w\Phi_{\vec{v}}^\dagger\,\partial_{\overline{w}}\Phi_{\vec{v}}\right) \\ &= \pi\theta\,16 f(\vec{v})\ \mathrm{Tr}\left(\partial_w P\,\partial_{\overline{w}}P\right) \\ &= 8\pi f(\vec{v})\ \mathrm{Tr}\left(\left[c^\dagger, P\right][P, c]\right) \\ &= 8\pi f(\vec{v})\,q = f(\vec{v})\,E[\Phi_{\vec{0}}]\,, \end{aligned} \tag{6.11}$$

because the final trace is, by unitary equivalence, independent of the motion. We have introduced the "velocity factor"

$$f(\vec{v}) := \frac{1}{16}\rho\bar\rho\,\nu\bar\nu\left(\sqrt{\mu\bar\mu} + \sqrt{\mu\bar\mu}^{-1}\right)^2 = \frac{1}{\nu\bar\nu} = \cosh\eta\,\sin\varphi = \frac{\sqrt{1-\vec{v}^2}}{1-v_y^2}\,. \tag{6.12}$$

It is noteworthy that for motion in the y direction ($\varphi = \pi/2$) the "velocity factor" $\cosh\eta$ agrees with the relativistic contraction factor $1/\sqrt{1-\vec{v}^2} \geq 1$, but it is smaller than one for motion with $|v_x| > |v_y|\sqrt{1-v_y^2}$, and the energy may be made arbitrarily close to zero by a large "boost", since $\vec{v}^2 \to 1$ is equivalent to $\varphi \to 0$ or $\eta \to \pm\infty$.

[2]For geometric visualization, the particular $\mathrm{SL}(2,\mathbb{R})$ transformation matrix appearing here may be factorized to $\begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix}\begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}$, where $2\delta = -\frac{\pi}{2} - \vartheta$.

Examples. We can immediately write down moving versions of our solutions (5.14) and (5.15) from the previous section. Since $|0\rangle' := U(t)|0\rangle$ denotes the moving ground state (i.e. squeezed and shifted), the $r=1$ abelian soliton reads

$$\begin{aligned}\Phi = 1 - \bar\rho\, U(t)\,|0\rangle\langle 0|\,U(t)^\dagger \Longrightarrow \Phi_\star &= 1 - 2\bar\rho\, e^{-w\overline{w}/\theta} \\ &= 1 - 2\bar\rho\, e^{-[\vec r - \vec v t]^T \Lambda^T \Lambda\, [\vec r - \vec v t]\,/\theta}\,,\end{aligned} \tag{6.13}$$

where Λ is the "squeezing matrix" appearing in (6.7). In a similar vein, the $r=1$ non-abelian solution (5.15) gets generalized to

$$\begin{aligned}\Phi &= 1 - \bar\rho\, U(t) \begin{pmatrix} \frac{\lambda\bar\lambda}{\lambda\bar\lambda + a^\dagger a} & \frac{\lambda}{\lambda\bar\lambda + a^\dagger a} a^\dagger \\ a \frac{\bar\lambda}{\lambda\bar\lambda + a^\dagger a} & a \frac{1}{\lambda\bar\lambda + a^\dagger a} a^\dagger \end{pmatrix} U(t)^\dagger \\ &= 1 - \bar\rho \begin{pmatrix} \frac{\lambda\bar\lambda}{\lambda\bar\lambda + c^\dagger c} & \frac{\lambda}{\lambda\bar\lambda + c^\dagger c} c^\dagger \\ c \frac{\bar\lambda}{\lambda\bar\lambda + c^\dagger c} & c \frac{1}{\lambda\bar\lambda + c^\dagger c} c^\dagger \end{pmatrix},\end{aligned} \tag{6.14}$$

which leads to

$$\Phi_\star = 1 - \bar\rho \begin{pmatrix} \frac{\gamma\overline{\gamma}}{\gamma\overline{\gamma} + w\overline{w} - \theta} & \gamma \frac{\gamma\overline{\gamma} + w\overline{w} - 2\theta}{(\gamma\overline{\gamma} + w\overline{w} - \theta)^2} \overline{w} \\ w \frac{\gamma\overline{\gamma} + w\overline{w} - 2\theta}{(\gamma\overline{\gamma} + w\overline{w} - \theta)^2} \overline{\gamma} & \frac{w\overline{w} + \theta}{\gamma\overline{\gamma} + w\overline{w} + \theta} \end{pmatrix}, \tag{6.15}$$

with the time dependence hiding in

$$\begin{aligned} w &= w(t,x,y) \\ &= \left(\cosh\tau - e^{i\vartheta}\sinh\tau\right)(x - v_x t) + i\left(\cosh\tau + e^{i\vartheta}\sinh\tau\right)(y - v_y t)\,, \end{aligned} \tag{6.16}$$

as expected. We note that in the distinguished frame the large-time limit (for $\vec v \neq \vec 0$) is

$$\lim_{|t|\to\infty} \Phi_\star = 1 - \bar\rho\,\Pi\,, \qquad \text{with} \qquad \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.17}$$

Although a one-soliton configuration with topological charge $q > 1$ may feature several separated lumps in its energy density it does not deserve to be termed a "multi-soliton", because those lumps do not display *relative* motion. True multi-solitons reduce to a collection of solitons only in the asymptotic regime ($|t| \to \infty$) or for large relative speed ($v_{\mathrm{rel}} \to \infty$); they will be investigated next.

7. Multi-soliton configurations

Two-soliton configurations. In section 4 we gave an explicit recipe how to construct multi-soliton configurations

$$\Phi^\dagger = 1 - \sum_{k=1}^{m} \frac{R_k}{\mu_k} \tag{7.1}$$

via (4.11), (4.15) and (4.27). It is known from the non-abelian commutative situation [28] that for polynomial functions $T_k(c_k)$ such solutions have finite energy and describe m different lumps of energy moving in the xy plane with distinct velocities $\vec{v}_k$ parametrized by $\mu_k = e^{\eta_k - i\varphi_k}$, in other words: multi-solitons. Since the explicit form of such configurations is more complicated than that of one-soliton solutions, we illustrate the generic features on the simplest examples, namely two-soliton configurations with $r=1$. Thus, in the following, $m=2$ and $k=1,2$.

We begin by solving $\Gamma^{\ell p}$ in terms of $\widetilde{\Gamma}_{pk}$ via (4.15), taking into account the noncommutativity:

$$\Gamma^{11} = +\Big[\widetilde{\Gamma}_{11} - \widetilde{\Gamma}_{12}\widetilde{\Gamma}_{22}^{-1}\widetilde{\Gamma}_{21}\Big]^{-1}, \qquad \Gamma^{12} = -\Big[\widetilde{\Gamma}_{11} - \widetilde{\Gamma}_{12}\widetilde{\Gamma}_{22}^{-1}\widetilde{\Gamma}_{21}\Big]^{-1}\,\widetilde{\Gamma}_{12}\widetilde{\Gamma}_{22}^{-1},$$
$$\Gamma^{21} = -\Big[\widetilde{\Gamma}_{22} - \widetilde{\Gamma}_{21}\widetilde{\Gamma}_{11}^{-1}\widetilde{\Gamma}_{12}\Big]^{-1}\,\widetilde{\Gamma}_{21}\widetilde{\Gamma}_{11}^{-1}, \qquad \Gamma^{22} = +\Big[\widetilde{\Gamma}_{22} - \widetilde{\Gamma}_{21}\widetilde{\Gamma}_{11}^{-1}\widetilde{\Gamma}_{12}\Big]^{-1}, \tag{7.2}$$

which in terms of T_1 and T_2 reads

$$\begin{aligned}
\Gamma^{11} &= +\left[\frac{1}{\mu_{11}}\,T_1^\dagger T_1 - \frac{\mu_{22}}{\mu_{12}\mu_{21}}\,T_1^\dagger T_2\frac{1}{T_2^\dagger T_2}T_2^\dagger T_1\right]^{-1}, \\
\Gamma^{12} &= -\left[\frac{1}{\mu_{11}}\,T_1^\dagger T_1 - \frac{\mu_{22}}{\mu_{12}\mu_{21}}\,T_1^\dagger T_2\frac{1}{T_2^\dagger T_2}T_2^\dagger T_1\right]^{-1}\frac{\mu_{22}}{\mu_{12}}\,T_1^\dagger T_2\frac{1}{T_2^\dagger T_2}, \\
\Gamma^{21} &= -\left[\frac{1}{\mu_{22}}\,T_2^\dagger T_2 - \frac{\mu_{11}}{\mu_{21}\mu_{12}}\,T_2^\dagger T_1\frac{1}{T_1^\dagger T_1}T_1^\dagger T_2\right]^{-1}\frac{\mu_{11}}{\mu_{21}}\,T_2^\dagger T_1\frac{1}{T_1^\dagger T_1}, \\
\Gamma^{22} &= +\left[\frac{1}{\mu_{22}}\,T_2^\dagger T_2 - \frac{\mu_{11}}{\mu_{21}\mu_{12}}\,T_2^\dagger T_1\frac{1}{T_1^\dagger T_1}T_1^\dagger T_2\right]^{-1},
\end{aligned} \tag{7.3}$$

with $\mu_{k\ell} := \mu_k - \bar{\mu}_\ell$. For the full solution, we then arrive at

$$\begin{aligned}
\Phi^\dagger &= 1 - \frac{T_1\Gamma^{11}T_1^\dagger}{\mu_1} - \frac{T_1\Gamma^{12}T_2^\dagger}{\mu_2} - \frac{T_2\Gamma^{21}T_1^\dagger}{\mu_1} - \frac{T_2\Gamma^{22}T_2^\dagger}{\mu_2} \\
&= 1 - \frac{\mu_{11}}{\mu_1}\,T_1\Big[T_1^\dagger(1-\sigma P_2)T_1\Big]^{-1}\,T_1^\dagger + \frac{\mu_{21}}{\mu_2}\sigma\,T_1\Big[T_1^\dagger(1-\sigma P_2)T_1\Big]^{-1}\,T_1^\dagger P_2 + \\
&\quad + \frac{\mu_{12}}{\mu_1}\sigma\,T_2\Big[T_2^\dagger(1-\sigma P_1)T_2\Big]^{-1}\,T_2^\dagger P_1 - \frac{\mu_{22}}{\mu_2}\,T_2\Big[T_2^\dagger(1-\sigma P_1)T_2\Big]^{-1}\,T_2^\dagger
\end{aligned} \tag{7.4}$$
$$\begin{aligned}
&= 1 - \frac{1}{1-\sigma P_1P_2}\left\{\frac{\mu_{11}}{\mu_1}\,P_1 - \frac{\mu_{21}}{\mu_2}\sigma\,P_1P_2\right\} - \\
&\quad - \frac{1}{1-\sigma P_2P_1}\left\{\frac{\mu_{22}}{\mu_2}\,P_2 - \frac{\mu_{12}}{\mu_1}\sigma\,P_2P_1\right\},
\end{aligned} \tag{7.5}$$

where $\sigma := \mu_{11}\mu_{22}/\mu_{12}\mu_{21} \in \mathbb{R}$, the T_k must be chosen to satisfy (4.27), and $P_k = T_k\frac{1}{T_k^\dagger T_k}T_k^\dagger$ as usual.

The abelian rank-one two-soliton is included here: one simply writes

$$T_k = |z_k,t\rangle = e^{z_k c_k^\dagger - \bar{z}_k c_k}\,|0\rangle_k = e^{z_k c_k^\dagger - \bar{z}_k c_k}\,U_k(t)\,|0\rangle = U_k(t)\,e^{z_k a^\dagger - \bar{z}_k a}\,|0\rangle\,. \tag{7.6}$$

Thus, non-commutative abelian fields look like commutative $\mathrm{U}(\infty)$ fields. Our kets are normalized to unity, and therefore $P_k = |z_k,t\rangle\langle z_k,t|$. In the bra-ket formalism our solution takes the form

$$\Phi^{\dagger} = 1 - \frac{1}{1-\sigma|s|^2}\left\{\frac{\mu_{11}}{\mu_1}\,|z_1,t\rangle\langle z_1,t| - \frac{\mu_{21}}{\mu_2}\sigma s\,|z_1,t\rangle\langle z_2,t| - \right.$$
$$\left. - \frac{\mu_{12}}{\mu_1}\sigma\bar{s}\,|z_2,t\rangle\langle z_1,t| + \frac{\mu_{22}}{\mu_2}\,|z_2,t\rangle\langle z_2,t|\right\}, \qquad (7.7)$$

where the overlap $\langle z_1,t|z_2,t\rangle$ was denoted by s.

The simplest non-abelian example is a U(2) configuration with $T_1 = \binom{1}{a}$ and $T_2 = \binom{1}{c} = UT_1U^{\dagger}$, i.e. taking $\mu_1 = -i$ and $\mu_2 =: \mu$. We refrain from writing down the lengthy explicit expression which results from inserting this into (7.3) and (7.4) or from plugging the projectors $P_k = T_k \frac{1}{T_k^{\dagger}T_k} T_k^{\dagger}$ into (7.5).

High-speed asymptotics. The two-soliton configuration (7.4) does not factorize into two one-soliton solutions, except in limiting situations, such as large relative speed or large time. Let us first consider the limit where the second lump is moving with the velocity of light,

$$\varphi \longrightarrow 0\,, \qquad \text{or} \qquad \pi \Longleftrightarrow \text{Im}\;\mu_2 \longrightarrow 0 \Longleftrightarrow |v_2| \longrightarrow 1\,, \qquad (7.8)$$

while the velocity of the first lump is subluminal. In this limit, we have $\sigma \to 0$ and therefore

$$\Phi^{\dagger} \longrightarrow 1 - \frac{\mu_{11}}{\mu_1}\,P_1 + \frac{\mu_{21}}{\mu_2}\sigma\,P_1P_2 + \frac{\mu_{12}}{\mu_1}\sigma\,P_2P_1 - \frac{\mu_{22}}{\mu_2}\,P_2\,. \qquad (7.9)$$

Define also $\rho_k := \mu_{kk}/\mu_k = 1 - \bar{\mu}_k/\mu_k$. Clearly, Im $\mu_2 \to 0$ implies that

$$\Phi^{\dagger} \longrightarrow 1 - \rho_1\,P_1\,, \qquad (7.10)$$

except for $\vec{v}_2 \to (0,\pm 1)$ meaning $\mu_2 \to 0$ or ∞, where

$$\Phi^{\dagger} \longrightarrow (1-\rho_1P_1)(1-\rho_2P_2)\,, \qquad \text{or} \qquad \Phi^{\dagger} \longrightarrow (1-\rho_2P_2)(1-\rho_1P_1)\,, \qquad (7.11)$$

respectively.

Large-time asymptotics. Now we investigate the behavior of (7.4) for large (positive and negative) time. For convenience, we take the first lump to be at rest while the second one moves, i.e.

$$\mu_1 = -i\,, \qquad \text{and} \qquad \mu_2 =: \mu \Longrightarrow \rho_1 = 2\,, \qquad \rho_2 =: \rho\,,$$
$$\sigma \;=\; 1 - \frac{(\bar{\mu}-i)(\mu+i)}{(\mu-i)(\bar{\mu}+i)}\,. \qquad (7.12)$$

For simplicity, we consider $r=1$, i.e. T_k is a column vector and P_k of (matrix) rank one. After some tedious algebra (see appendix B) one learns that $\Phi^{\dagger}$ factorizes in the large-time limit:

$$\Phi^{\dagger} \longrightarrow \left(1 - 2\,\widetilde{P}_1\right)(1-\rho\,\Pi_2)\,, \qquad (7.13)$$

where $\Pi_2 := \lim_{|t|\to\infty} P_2$ is a coordinate-independent hermitean projector, and

$$\widetilde{P}_1 = [\alpha\,\Pi_2 + (1-\Pi_2)]\,T_1\,\frac{1}{T_1^\dagger(1-\sigma\Pi_2)\,T_1}\,T_1^\dagger\,[\bar{\alpha}\,\Pi_2 + (1-\Pi_2)]\,,$$
$$\text{with} \qquad \alpha = \frac{\bar{\mu}-i}{\mu-i} \tag{7.14}$$

is a new projector describing a one-soliton configuration.

We have explicitly checked this behavior for the simple U(2) example mentioned above and found indeed

$$\Phi^\dagger(x,y,t \longrightarrow \pm\infty) = \left(1 - 2\,\widetilde{P}_1\right)(1-\rho\,\Pi_2) + O(t^{-1})\,,$$
$$\text{with} \qquad \widetilde{P}_1 = \widetilde{T}_1 \frac{1}{\widetilde{T}_1^\dagger \widetilde{T}_1} \widetilde{T}_1^\dagger\,, \tag{7.15}$$

where

$$\widetilde{T}_1 = \begin{pmatrix} 1 \\ \alpha\,a \end{pmatrix}, \qquad \text{and} \qquad \Pi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.16}$$

Let us now consider the situation within the frame moving with the second lump. In this frame the second lump is static. By the same arguments as above, the solution $\Phi^\dagger$ behaves in the large-time limit as

$$\Phi^\dagger \longrightarrow (1 - 2\,\Pi_1)\left(1 - \rho\,\widetilde{P}_2\right), \tag{7.17}$$

where Π_1 is a coordinate-independent hermitean projector, and the projector $\widetilde{P}_2$ describes the second lump in the moving frame. It is obvious how these statements generalize to m-soliton configurations. If we sit in the frame moving with the ℓth lump, then the large-time limit of $\Phi^\dagger$ is a product of $(1-\rho_\ell\widetilde{P}_\ell)$ and constant unitary matrices,

$$\begin{aligned}\Phi^\dagger \longrightarrow\ & (1-\rho_1\Pi_1)\,(1-\rho_2\Pi_2)\cdots \\ & \cdots(1-\rho_{\ell-1}\Pi_{\ell-1})\left(1-\rho_\ell\widetilde{P}_\ell\right)(1-\rho_{\ell+1}\Pi_{\ell+1})\cdots(1-\rho_m\Pi_m)\,.\end{aligned} \tag{7.18}$$

The proof is outlined in appendix C.

A lesson from (7.18) concerns the scattering properties of our multi-soliton solutions. Since the limits $t\to-\infty$ and $t\to+\infty$ in (7.18) are identical, every lump escapes completely unharmed from the encounter with the other ones. The absence of any change in velocity, shape, or displacement evidences the no-scattering feature of these lumps, be they commutative or not. Summarizing, what survives at $|t|\to\infty$ is at most a one-soliton configuration with modified parameters and multiplied with constant unitary matrices.

The asymptotic study just performed covers the U(1) case as well, with some additional qualifications. At rank one and $m = 2$, the static lump is parametrized by

$$|z_1, 0\rangle \equiv |z_1\rangle = e^{z_1 a^\dagger - \bar{z}_1 a}\,|0\rangle\,,$$

$$\text{while} \qquad |z_2, t\rangle = e^{z_2 c^\dagger - \bar{z}_2 c}\,U(t)\,|0\rangle = U(t)\,e^{z_2 a^\dagger - \bar{z}_2 a}\,|0\rangle \tag{7.19}$$

representing the moving lump remains time-dependent as $|t| \to \infty$. Yet, the analysis is much simpler than for the non-abelian case. Since [39]

$$\langle z_1, 0|z_2, t\rangle = e^{-\frac{1}{2}|\beta_2|^2\,t^2}\left(1 - \frac{1}{2}|\xi_2|^2\right), \tag{7.20}$$

the overlap $s = \langle z_1, 0|z_2, t\rangle$ goes to zero exponentially fast and the solitons become well separated. Therefore, (7.7) tends to

$$\Phi^\dagger \longrightarrow 1 - 2\,|z_1\rangle\,\langle z_1| - \rho\,|z_2, t\rangle\,\langle z_2, t|\,, \tag{7.21}$$

which decomposes additively (rather than multiplicatively) into two independent solitons.

Energies. It is technically difficult to compute the energy of true multi-solution configurations such as (7.4). However, because energy is a conserved quantity, we may evaluate it in the limit of $|t| \to \infty$. From the result (7.18) we seem to infer that generically only the ℓth soliton contributes to the total energy. Yet, this is incorrect, as can be seen already in the commutative case: even though the moving lumps have disappeared from sight at $|t| = \infty$, their energies have not, because the integral of the energy density extends to spatial infinity, and so it is not legitimate to perform the large-time limit before the integration. What (7.18) shows, however, is the large-time vanishing of the *overlap* between the lumps in relative motion, so that asymptotically the lumps pertaining to different values of k are well separated. The total energy, being a local functional, then approaches a sum of m contributions, each of which stems from a group of r lumps, in isolation from the other groups.

To compute the ℓth contribution, we must ignore the influence of the other groups of lumps. To this end we take the large-time limit and treat the ℓth group as a moving one-soliton. From (6.11) we know that its energy is

$$E_\ell(\vec{v}_\ell) = f(\vec{v}_\ell)\,E_\ell\left(\vec{0}\right) = 8\pi\,f(\vec{v}_\ell)\,q_\ell\,. \tag{7.22}$$

By adding all contributions, it follows that the total energy of the m-soliton solution at any time is

$$E = 8\pi \sum_{k=1}^{m} q_k\,\cosh\eta_k\,\sin\varphi_k\,. \tag{7.23}$$

8. Conclusions

In this paper we have demonstrated that the power of integrability can be extended to non-commutative field theories without problems. For the particular case of a $2+1$-dimensional integrable sigma model which captures the BPS sector of the $2+1$-dimensional Yang-Mills-Higgs system, the "dressing method" was applied to generate a wide class of multi-soliton solutions to the non-commutative field equations. We were able to transcend the limitations of standard non-commutative scalar and gauge theories, by providing a general construction scheme for and several examples of explicit analytical multi-soliton configurations, with *full time dependence*, for *finite* θ values, and for any $\mathrm{U}(n)$ *gauge group*. Moreover, two proofs of large-time asymptotic factorization into a product of single solitons were presented, and the energies and topological charges have been computed.

The model considered here does not stand alone, but is motivated by string theory. As was explained in two previous papers [27, 41], n coincident D2-branes in a 2+2-dimensional space-time give rise to open $N=2$ string dynamics [42]–[46] which (on tree level[3]) is completely described by our integrable $\mathrm{U}(n)$ sigma model living on the brane. In this context, the multi-solitons are to be regarded as D0-branes moving inside the D2-brane(s). Switching on a constant NS B-field and taking the Seiberg-Witten decoupling limit deforms the sigma model noncommutatively, admitting also regular abelian solitons.

Since the massless mode of the open $N=2$ string in a space-time-filling brane parametrizes self-dual Yang-Mills in $2+2$ dimensions [42], it is not surprising that the sigma-model field equations also derive from the self-duality equations by dimensional reduction [28]. Moreover, it has been shown that most (if not all) integrable equations in three and less dimensions can be obtained from the self-dual Yang-Mills equations (or their hierarchy) by suitable reductions (see e.g. [49]–[53] and references therein). This implies, in particular, that open $N=2$ strings can provide a consistent quantization of integrable models in $1 \leq D \leq 3$ dimensions. Adding a constant B-field background yields *non-commutative* self-dual Yang-Mills on the world volumes of coincident D3-branes and, hence, our *non-commutative* modified sigma model on the world volumes of coincident D2-branes [41]. It is certainly of interest to study also other reductions of the non-commutative self-dual Yang-Mills equations to non-commutative versions of KdV, non-linear Schrödinger or other integrable equations, as have been considered e.g. in [20, 22, 23, 54].

Another feature of these models is a very rich symmetry structure. All symmetries of our equations can be derived (after non-commutative deformation) from symmetries of the self-dual Yang-Mills equations [55, 56] or from their stringy generalizations [38, 57]. For this one has to extend the relevant infinitesimal symmetries and integrable hierarchies to the non-commutative case. In fact, such symmetries can

[3] for a one-loop analysis see [47, 48]

be realized as derivatives w.r.t. moduli parameters entering the solutions described in sections 5 to 7 or, in a more general case, as vector fields on the moduli space.

In this paper we considered the simultaneous and relative motion of several non-commutative lumps of energy. Like in the commutative limit, it turned out that *no scattering* occurs for the non-commutative solitons within the ansatz considered here. This means that the different lumps of energy move linearly in the spatial plane with transient shape deformations at mutual encounters but asymptotically emerge without time delay or changing profile or velocity. However, starting from the ansatz (4.9) containing *second-order* poles in ζ, one can actually construct explicit solutions featuring *non-trivial scattering* of non-commutative solitons. Such configurations will generalize earlier results obtained for commutative solitons [58]–[61] of the modified sigma model in $2+1$ dimensions.

It would also be very interesting to construct the Seiberg-Witten map for our soliton configurations along the lines presented in [62]. This should allow one to compare their representation in commutative variables with their profile in the star-product formulation.

Finally, an exciting relation between the algebraic structures of non-commutative soliton physics and string field theory has surfaced recently [63, 64, 65]. It is tempting to speculate that the "sliver states" will play a role in the analysis of the open $N=2$ string field theory [57, 66]. We intend to investigate such questions in the near future.

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A. Coherent and squeezed states

We collect some definitions and useful relations pertaining to coherent and squeezed states [39].

Coherent states for the Heisenberg algebra $[a, a^\dagger] = 1$ are generated by the unitary shift operator

$$D(\beta) := e^{\beta a^\dagger - \bar{\beta} a} = e^{-\frac{1}{2}\bar{\beta}\beta}\, e^{\beta a^\dagger}\, e^{\bar{\beta} a}\,. \tag{A.1}$$

It acts on the Heisenberg algebra as a shift,

$$D \begin{pmatrix} a \\ a^\dagger \end{pmatrix} D^\dagger = \begin{pmatrix} a \\ a^\dagger \end{pmatrix} - \begin{pmatrix} \beta \\ \bar{\beta} \end{pmatrix}, \tag{A.2}$$

and operates on the Fock space via

$$
\begin{aligned}
D(\beta)\,|0\rangle &= e^{-\frac{1}{2}\bar{\beta}\beta}\, e^{\beta a^\dagger}\,|0\rangle\,, \\
\langle 0|\, D^\dagger(\beta_1)\, D(\beta_2)\,|0\rangle &= e^{-\frac{1}{2}\bar{\beta}_1\beta_1}\, e^{-\frac{1}{2}\bar{\beta}_2\beta_2}\, e^{\bar{\beta}_1\beta_2}\,.
\end{aligned} \tag{A.3}
$$

A representation of $\mathrm{su}(1,1)$ can be constructed in terms of bilinears in the Heisenberg algebra,

$$
K_+ := \frac{1}{2} a^{\dagger 2}\,, \qquad K_- := \frac{1}{2} a^2\,, \qquad K_0 := \frac{1}{4}\left(a a^\dagger + a^\dagger a\right), \tag{A.4}
$$

which fulfill

$$
[K_0, K_\pm] = \pm K_\pm\,, \qquad \text{and} \qquad [K_+, K_-] = -2K_0\,. \tag{A.5}
$$

The unitary squeezing operator S may be defined as

$$
S(\alpha) := e^{\alpha K_+ - \bar{\alpha} K_-} = e^{\xi K_+}\, e^{-\ln(1-\bar{\xi}\xi) K_0}\, e^{-\bar{\xi} K_-} =: S(\xi)\,, \tag{A.6}
$$

where

$$
\alpha =: e^{i\vartheta}\,\tau\,, \qquad \text{and} \qquad \xi = e^{i\vartheta}\,\tanh\tau\,. \tag{A.7}
$$

S induces a representation on the Heisenberg algebra via

$$
S \begin{pmatrix} a \\ a^\dagger \end{pmatrix} S^\dagger = g \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \qquad \text{with} \qquad g = \begin{pmatrix} \cosh\tau & -e^{i\vartheta}\sinh\tau \\ -e^{-i\vartheta}\sinh\tau & \cosh\tau \end{pmatrix}. \tag{A.8}
$$

The action of S on the Fock space is characterized by

$$
\begin{aligned}
S(\xi)\,|0\rangle &= \left(1-\bar{\xi}\xi\right)^{1/4} e^{\xi K_+}\,|0\rangle\,, \\
\langle 0|\, S^\dagger(\xi_1)\, S(\xi_2)\,|0\rangle &= \left(1-\bar{\xi}_1\xi_1\right)^{1/4}\left(1-\bar{\xi}_2\xi_2\right)^{1/4}\left(1-\bar{\xi}_1\xi_2\right)^{-1/2}.
\end{aligned} \tag{A.9}
$$

The combined action of shifting and squeezing leads to squeezed states of the form

$$
|\xi,\beta\rangle := S(\xi)\, D(\beta)\,|0\rangle = e^{-\frac{1}{2}\bar{\beta}\beta}\left(1-\bar{\xi}\xi\right)^{1/4} e^{\frac{1}{2}\xi a^{\dagger 2} + \beta a^\dagger}\,|0\rangle\,, \tag{A.10}
$$

which are normalized to unity due to the unitarity of S and D. The corresponding transformation of the Heisenberg generators reads

$$
\begin{pmatrix} c \\ c^\dagger \end{pmatrix} := S\, D \begin{pmatrix} a \\ a^\dagger \end{pmatrix} D^\dagger\, S^\dagger = \begin{pmatrix} \cosh\tau & -e^{i\vartheta}\sinh\tau \\ -e^{-i\vartheta}\sinh\tau & \cosh\tau \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} - \begin{pmatrix} \beta \\ \bar{\beta} \end{pmatrix}, \tag{A.11}
$$

where the order of action was relevant. It is obvious that the unitary transformations defined in (4.23) are generated by $U_k(t) = S(\xi_k) D(\beta_k t)$ with parameters $\{\xi_k, \beta_k\}$ determined by μ_k.

B. Asymptotic factorization for the two-soliton configuration

We set out to proof the result (7.13). With $\mu_1 = -i$ and $\mu_2 = \mu \neq -i$, we first consider the distinguished $(z, \bar{z}, t)$ coordinate frame. Let the entries of T_2 be polynomials of (the same) degree q_2 in c_2. At large times, (4.23) implies that $c_2 \to -\beta_2\, t$, and the t^{q_2} term in each polynomial will dominate,

$$T_2 \longrightarrow t^{q_2} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} =: t^{q_2}\, \vec{\gamma}\,, \tag{B.1}$$

where $\vec{\gamma}$ is a fixed vector in group space. Due to the homogeneity of (7.4), the proportionality factor t^{q_2} can be dropped and $\vec{\gamma}$ be normalized to unity, $\vec{\gamma}^{\,\dagger}\vec{\gamma} = 1$. We may then substitute

$$T_2^\dagger\, T_2 \longrightarrow 1\,, \qquad \text{and} \qquad P_2 \longrightarrow T_2\, T_2^\dagger \longrightarrow \vec{\gamma}\;\vec{\gamma}^{\,\dagger} =: \Pi = \Pi^\dagger\,, \tag{B.2}$$

the projector onto the $\vec{\gamma}$ direction. Note that the entries of the constant matrix Π are commuting. Furthermore, we introduce $p := \vec{\gamma}^{\,\dagger} P_1\, \vec{\gamma}$ which is a group scalar but noncommuting. Performing these substitutions in (7.4), the two-soliton configuration becomes

$$\begin{aligned}\Phi^\dagger \longrightarrow\; & 1 - \frac{\mu_{11}}{\mu_1}\, T_1\, \frac{1}{T_1^\dagger(1-\sigma\Pi)T_1}\, T_1^\dagger + \frac{\mu_{21}}{\mu_2}\sigma\, T_1\, \frac{1}{T_1^\dagger(1-\sigma\Pi)T_1}\, T_1^\dagger\, \Pi\, + \\ & + \frac{\mu_{12}}{\mu_1}\sigma\, \frac{1}{1-\sigma p}\, \Pi\, P_1 - \frac{\mu_{22}}{\mu_2}\, \frac{1}{1-\sigma p}\, \Pi\,. \end{aligned} \tag{B.3}$$

In order to turn the terms in the first line to projectors, it is useful to introduce a constant matrix

$$M = \alpha\,\Pi + (1-\Pi)\,, \qquad \text{with} \qquad \alpha = \frac{\bar{\mu}-i}{\mu-i}\,, \qquad \text{so that} \qquad M^\dagger M = 1 - \sigma\Pi \tag{B.4}$$

and rewrite

$$M\, T_1 =: \widetilde{T}\,, \qquad \text{so that} \qquad T_1\, \frac{1}{T_1^\dagger M^\dagger M\, T_1}\, T_1^\dagger \;=\; \frac{1}{M}\widetilde{P}\frac{1}{M^\dagger}\,, \qquad \text{with} \qquad \widetilde{P} := \widetilde{T}\, \frac{1}{\widetilde{T}^\dagger \widetilde{T}}\, \widetilde{T}^\dagger\,. \tag{B.5}$$

We thus arrive at

$$\Phi^\dagger \longrightarrow 1 - \frac{\mu_{11}}{\mu_1}\, \frac{1}{M}\widetilde{P}\frac{1}{M^\dagger} + \frac{\mu_{21}}{\mu_2}\sigma\, \frac{1}{M}\widetilde{P}\frac{1}{\bar{\alpha}}\,\Pi + \frac{\mu_{12}}{\mu_1}\sigma\, \frac{1}{1-\sigma p}\, \Pi\, P_1 - \frac{\mu_{22}}{\mu_2}\, \frac{1}{1-\sigma p}\, \Pi\,, \tag{B.6}$$

after making use of

$$\begin{aligned} & f(M)\,\Pi = \Pi\, f(M) = f(\alpha)\,\Pi\,, \\ \text{but} \qquad & f(M)\,(1-\Pi) = (1-\Pi)\, f(M) = 1-\Pi\,. \end{aligned} \tag{B.7}$$

Further analysis is facilitated by decomposing $\Phi^\dagger$ into parts orthogonal and parallel to $\vec{\gamma}$. We insert our values (7.12) for the coefficients and find

$$
\begin{aligned}
(1-\Pi)\,\Phi^\dagger\,(1-\Pi) &\longrightarrow (1-\Pi)\left(1-2\,\widetilde{P}\right)(1-\Pi)\,, \\
(1-\Pi)\,\Phi^\dagger\,\vec{\gamma} &\longrightarrow (1-\Pi)\left(-2\,\widetilde{P}\,\frac{1}{\bar{\alpha}}+\frac{2i(\mu-\bar{\mu})}{\mu(\bar{\mu}+i)}\,\widetilde{P}\,\frac{1}{\bar{\alpha}}\right)\vec{\gamma}\,, \\
\vec{\gamma}^{\,\dagger}\,\Phi^\dagger\,(1-\Pi) &\longrightarrow \vec{\gamma}^{\,\dagger}\left(-2\,\frac{1}{\alpha}\,\widetilde{P}+\frac{2(\mu-\bar{\mu})}{\mu-i}\,\frac{1}{1-\sigma p}\,P_1\right)(1-\Pi)\,, \\
\vec{\gamma}^{\,\dagger}\,\Phi^\dagger\,\vec{\gamma} &\longrightarrow \vec{\gamma}^{\,\dagger}\left(1-2\,\frac{1}{\alpha}\,\widetilde{P}\,\frac{1}{\bar{\alpha}}+\frac{2i(\mu-\bar{\mu})}{\mu(\bar{\mu}+i)}\,\frac{1}{\alpha}\,\widetilde{P}\,\frac{1}{\bar{\alpha}}+\right. \\
&\qquad\left.+\frac{2(\mu-\bar{\mu})}{\mu-i}\,\frac{p}{1-\sigma p}-\frac{\mu-\bar{\mu}}{\mu}\,\frac{1}{1-\sigma p}\right)\vec{\gamma}\,.
\end{aligned}
\tag{B.8}
$$

With the help of the identities

$$
\vec{\gamma}^{\,\dagger}\,\widetilde{P}\,\vec{\gamma}=\alpha\,\vec{\gamma}^{\,\dagger}\,\frac{1}{1-\sigma P_1\Pi}\,P_1\,\vec{\gamma}\,\bar{\alpha}=\alpha\,\bar{\alpha}\,\frac{p}{1-\sigma p}\,, \tag{B.9}
$$

$$
\text{and}\qquad \vec{\gamma}^{\,\dagger}\,\widetilde{P}\,(1-\Pi)=\alpha\,\vec{\gamma}^{\,\dagger}\,\frac{1}{1-\sigma P_1\Pi}\,P_1\,(1-\Pi)=\alpha\,\frac{1}{1-\sigma p}\,\vec{\gamma}^{\,\dagger}\,P_1\,(1-\Pi)
$$

plus some algebra, the pieces of $\Phi^\dagger$ in (B.8) simplify to

$$
\begin{aligned}
(1-\Pi)\,\Phi^\dagger\,(1-\Pi) &\longrightarrow (1-\Pi)\left(1-2\,\widetilde{P}\right)(1-\Pi)\,, \\
(1-\Pi)\,\Phi^\dagger\,\vec{\gamma} &\longrightarrow (1-\Pi)\left(-2\,\widetilde{P}+2\,\rho\,\widetilde{P}\right)\vec{\gamma}\,, \\
\vec{\gamma}^{\,\dagger}\,\Phi^\dagger\,(1-\Pi) &\longrightarrow \vec{\gamma}^{\,\dagger}\left(-2\,\widetilde{P}\right)(1-\Pi)\,, \\
\vec{\gamma}^{\,\dagger}\,\Phi^\dagger\,\vec{\gamma} &\longrightarrow \vec{\gamma}^{\,\dagger}\left(1-\rho-2\,\widetilde{P}+2\,\rho\,\widetilde{P}\right)\vec{\gamma}\,,
\end{aligned}
\tag{B.10}
$$

which proves the factorization[4]

$$
\Phi^\dagger \longrightarrow \left(1-2\,\widetilde{P}\right)(1-\rho\,\Pi)\,. \tag{B.11}
$$

At $|t|\to\infty$ we are thus left with a static one-soliton configuration with modified parameters ($T_1\to MT_1$) and multiplied with a constant matrix $(1-\rho\Pi)$. If we move with the *second* lump the proof is the same, because then $a\to\beta_2 t$ and P_1 (in place of P_2) goes to a constant projector.

C. Asymptotic form of m-soliton configurations

The dressing method may be used to show the large-time factorization of multi-soliton solutions into a product of one-soliton solutions directly on the level of the auxiliary function $\psi(t,x,y,\zeta)$.

[4] We have omitted writing "soliton indices" for $\widetilde{P}$ and Π, to avoid cluttering our formulae.

Starting from the one-soliton ansatz,

$$\psi_1 = 1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1}\, \widehat{P}_1 \tag{C.1}$$

with a hermitean projector $\widehat{P}_1$, we may try to obtain a two-soliton solution by "dressing" the latter with another factor of the same kind ($\mu_1 \neq \mu_2$),

$$\psi_2 = \left(1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1}\, \widehat{P}_1\right) \left(1 + \frac{\mu_2 - \bar{\mu}_2}{\zeta - \mu_2}\, \widehat{P}_2\right). \tag{C.2}$$

The reality condition (4.2) is identically fulfilled as soon as $\widehat{P}_1$ and $\widehat{P}_2$ are hermitean projectors,

$$\widehat{P}_1 = \widehat{T}_1\, \frac{1}{\widehat{T}_1^\dagger \widehat{T}_1}\, \widehat{T}_1^\dagger\,, \qquad \text{and} \qquad \widehat{P}_2 = \widehat{T}_2\, \frac{1}{\widehat{T}_2^\dagger \widehat{T}_2}\, \widehat{T}_2^\dagger\,. \tag{C.3}$$

The removability of the pole for $\zeta \to \bar{\mu}_2$ in (4.8) is guaranteed if

$$\left(1 - \widehat{P}_2\right) c_2\, \widehat{P}_2 = 0 \Longrightarrow c_2\, \widehat{T}_2 = \widehat{T}_2\, Z_2\,, \tag{C.4}$$

qualifying the second factor on the r.h.s. of (C.2) as a standard one-soliton solution, as obtained from solving (4.28). We therefore redenote $\widehat{P}_2 \to \widetilde{P}_2$ and $\widehat{T}_2 \to \widetilde{T}_2$.[5] The vanishing of the $\zeta \to \bar{\mu}_1$ residue, on the other hand, leads to

$$(1 - \widehat{P}_1) \left(1 + \frac{\mu_2 - \bar{\mu}_2}{\bar{\mu}_1 - \mu_2}\, \widetilde{P}_2\right) c_1 \left(1 + \frac{\bar{\mu}_2 - \mu_2}{\bar{\mu}_1 - \bar{\mu}_2}\, \widetilde{P}_2\right) \widehat{P}_1 = 0\,, \tag{C.5}$$

which does not yield a "holomorphicity condition" for $\widehat{T}_1$ but rather demonstrates that $\widehat{P}_1$ is *not* a standard one-soliton projector.

Let us move with the first lump. In the $|t| \to \infty$ limit, then, the arguments used in (B.1) and (B.2) apply, meaning that $\widetilde{P}_2 \to \Pi_2$. Since Π_2 is a coordinate-independent projector, the c_1 in (C.5) can be moved next to $\widehat{P}_1$ which implies

$$\widetilde{P}_2 \longrightarrow \Pi_2 \Longrightarrow \left(1 - \widehat{P}_1\right) c_1\, \widehat{P}_1 = 0 \Longrightarrow c_1\, \widehat{T}_1 = \widehat{T}_1\, Z_1\,, \tag{C.6}$$

so that the large-time limit of $\widehat{P}_1$ is also a *standard* one-soliton projector, and we again redenote $\widehat{P}_1 \to \widetilde{P}_1$. Hence,

$$\lim_{|t|\to\infty} \psi_2 = \left(1 + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1}\, \widetilde{P}_1\right) \left(1 + \frac{\mu_2 - \bar{\mu}_2}{\zeta - \mu_2}\, \Pi_2\right), \tag{C.7}$$

which yields

$$\lim_{|t|\to\infty} \Phi_2^\dagger = \left(1 - \rho_1\, \widetilde{P}_1\right) (1 - \rho_2\, \Pi_2)\,. \tag{C.8}$$

[5] It will be seen shortly that these objects coincide with the ones defined in appendix B.

On the other hand, if we move with the second lump, then, in the large-time limit, $c_1 \to -(\beta_1 - \beta_2)t$, implying $\widehat{P}_1 \to \Pi_1$ where Π_1 is a coordinate-independent projector. In this limit, (C.5) is satisfied, and (C.4) shows that $\widehat{P}_2 \equiv \widetilde{P}_2$ describes a one-soliton configuration. Hence, in this frame we find

$$\lim_{|t|\to\infty} \Phi_2^\dagger = (1 - \rho_1\,\Pi_1)\left(1 - \rho_2\,\widetilde{P}_2\right). \tag{C.9}$$

It can be shown, in fact, that the multi-soliton ansatz (4.10) is equivalent to the product ansatz

$$\psi_m = \prod_{k=1}^{m} \frac{\mu_k - \bar{\mu}_k}{\zeta - \mu_k}\,\widehat{P}_k\,, \qquad \text{with} \qquad \widehat{P}_k = \widehat{T}_k\,\frac{1}{\widehat{T}_k^\dagger \widehat{T}_k}\,\widehat{T}_k^\dagger\,, \tag{C.10}$$

where in general the $\widehat{T}_k$ are "non-holomorphic" matrices. By induction of the above argument one easily arrives at the m-soliton generalization of (C.8). Namely, in the frame moving with the ℓth lump we have

$$\begin{aligned} \lim_{|t|\to\infty} \Phi_m^\dagger &= (1 - \rho_1\Pi_1)\cdots \\ &\cdots(1 - \rho_{\ell-1}\Pi_{\ell-1})\left(1 - \rho_\ell\widetilde{P}_\ell\right)(1 - \rho_{\ell+1}\Pi_{\ell+1})\cdots(1 - \rho_m\Pi_m)\,. \end{aligned} \tag{C.11}$$

This provides an alternative proof of large-time factorization.

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